ON SOME INDIRECT METHODS OF OBTAINING INFORMATION ON THE POSITION OF A CONTROLLED SYSTEM IN THE PHASE SPACE

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In designing optimal automatic control systems the control algorithm, determined by means of the methods of the theory of dynamic programming or in accordance with the maximum principle of Pontriagin, is realized on the basis of information on the instantaneous position of the controlled system in the phase space [1,2].

In numerous cases it is difficult to obtain such information, since not all the phase coordinates of the system may be accessible to measurement. It is frequently impossible to measure some phase coordinates due to absence of information on the position of the orientation system relative to which the position of the controlled system is to be determined. Thus, for instance, on a moving ship the direction of the real vertical of the location may be unknown relative to which the errors of the gyroscopic pendulum have to be determined or the direction of the geographical meridian relative to which the errors of the gyroscopic compass have to be determined.

In view of this, it is of interest to seek indirect methods of determination of the position of a controlled system in phase space. One of the possible methods is considered in this paper, applicable for linear stationary and nonstationary controlled systems.

1. Steady-state systems. The equations of motion of a steady-state controlled system can be written down in the following form:

$$\sum_{k=1}^{n} f_{jk}(D)\, y_k = x_j(t) \qquad (j = 1, \ldots, n) \tag{1.1}$$

In this expression, y_k are the generalized coordinates of the system and $x_j(t)$ are the external forces applied to the system. The functions $f_{jk}(D)$ are polynomials in D with constant coefficients and $D = d/dt$ where t is the time.

The set of equations (1.1) can be rewritten in the following form:

$$b_{j1}\, y_1^{(m_1)} + b_{j2} y_2^{(m_2)} + \ldots + b_{jn} y_n^{(m_n)} = \\ = \psi_j (y_1^{(m_1-1)}, \ldots, y_1, \ldots, y_n^{(m_n-1)}, \ldots, y_n) + x_j (t) \qquad (j = 1, \ldots, n) \tag{1.2}$$

Here the superscript (m_k) $(k = 1, \ldots, n)$ represents the order of the highest derivative y_k with respect to time which is found in (1.1). The functions ψ_j which enter into (1.2) will be linear functions of their arguments. Assuming that the determinant

$$\Delta^* = | b_{jk} | \tag{1.3}$$

has a non-zero value, the set of equations (1.2) can be solved for the highest derivatives $y_k^{(m_k)}$ with the result

$$y_j^{(m_j)} = F_j (y_1^{(m_1-1)}, \ldots, y_1, \ldots, y_n^{(m_n-1)}, \ldots, y_n) + \\ + \frac{B_{1j}}{\Delta^*} x_1 (t) + \ldots + \frac{B_{nj}}{\Delta^*} x_n (t) \qquad (j = 1, \ldots, n) \tag{1.4}$$

where F_j are linear functions of their arguments and B_{ij} are algebraic complements of the elements b_{ij} in the determinant (1.3). In order to transform the set of equations (1.4) into the Cauchy form, let us introduce the new variables

$$z_1 = y_1,\ z_2 = \dot{y}_1, \ldots, z_{m_1} = y_1^{(m_1-1)}, \ldots, z_r = y_n^{(m_n-1)} \tag{1.5}$$

where

$$r = m_1 + m_2 + \ldots + m_n \tag{1.6}$$

The new variables $z_1, \ldots, z_r$ represent the phase coordinates of the system. Furthermore, let us denote the linear combinations of external forces which enter into the right-hand sides of (1.4) by $X_{\sigma_j}(t)$, so that

$$X_{\sigma_j} (t) = \frac{B_{1j}}{\Delta^*} x_1 (t) + \ldots + \frac{B_{nj}}{\Delta^*} x_n (t) \qquad (\sigma_j = \sigma_1, \ldots, \sigma_n) \tag{1.7}$$

where

$$\sigma_1 = m_1, \qquad \sigma_2 = m_1 + m_2, \ldots, \sigma_n = r \tag{1.8}$$

Equations (1.4) can now be rewritten in the form

$$\dot{z}_1 - z_2 = 0, \ldots, \dot{z}_{m_1} - F_1(z_1, z_2, \ldots, z_r) = X_{\sigma_1}(t), \ldots,$$
$$\dot{z}_r - F_n(z_1, z_2, \ldots, z_r) = X_{\sigma_n}(t) \tag{1.9}$$

Since the functions $F_j(z_1, z_2, \ldots, z_r$ are linear functions of their arguments, it follows that Equations (1.9) can be rewritten in the form

$$\dot{z}_j + \sum_{k=1}^{r} a_{jk} z_k = X_j(t) \qquad (j = 1, \ldots, r) \tag{1.10}$$

We note that in (1.10)

$$X_\mu(t) \equiv 0 \text{ for } \mu \neq \sigma_l \qquad (l = 1, \ldots, n) \tag{1.11}$$

The set of scalar equations (1.10) is equivalent to the matrix equation

$$\dot{z} + az = X(t) \tag{1.12}$$

where

$$z = \|z_j\|, \qquad a = \|a_{jk}\|, \qquad X(t) = \|X_j(t)\| \tag{1.13}$$

The solution of (1.12) can be determined by operational methods. Assuming

$$\zeta(p) \div z(t), \qquad \Xi(p) \div X(t) \tag{1.14}$$

and bearing in mind the fact that $p\zeta(p) - pz(0) \div \dot{z}(t)$, we find in accordance with (1.12) that

$$\varphi(p)\zeta(p) = pz(0) + \Xi(p) \tag{1.15}$$

where

$$\varphi(p) = pE + a \tag{1.16}$$

and E denotes the unit matrix.

Denoting by $\Phi(p)$ the adjoint of the matrix $\phi(p)$ and by $\Delta(p)$ its determinant, we find from (1.15) that

$$\zeta(p) = \frac{p\Phi(p)}{\Delta(p)} z(0) + \frac{\Phi(p)\,\Xi(p)}{\Delta(p)} \tag{1.17}$$

Let us denote by $N(t)$ the original for the following representation:

$$\frac{p\Phi(p)}{\Delta(p)} \div N(t) \tag{1.18}$$

The function $N(t)$ in an $r \times r$ matrix

$$N(t) = \| N_{jk}(t) \| \tag{1.19}$$

The elements of the matrix $N(t)$ are of the form [3]

$$\begin{aligned} N_{jk}(t) = {\sum_{\sigma}}' \frac{e^{\varkappa_\sigma t}}{(q_\sigma - 1)!} \left[\left(t + \frac{\partial}{\partial p} \right)^{q_\sigma - 1} \frac{\Phi_{jk}(p)}{\Delta_\sigma(p)} \right]_{p=\varkappa_\sigma} + \\ + 2 {\sum_{h}}' \frac{e^{\varepsilon_h t}}{(q_{s'+h} - 1)!} \left\{ \mathrm{Re} \left[\left(t + \frac{\partial}{\partial p} \right)^{q_{s'+h} - 1} \frac{\Phi_{jk}(p)}{\Delta_{s'+h}(p)} \right]_{p=\varepsilon_h + i\omega_h} \cos \omega_h t - \right. \\ \left. - \mathrm{Im} \left[\left(t + \frac{\partial}{\partial p} \right)^{q_{s'+h} - 1} \frac{\Phi_{jk}(p)}{\Delta_{s'+h}(p)} \right]_{p=\varepsilon_h + i\omega_h} \sin \omega_h t \right\} \end{aligned} \tag{1.20}$$

Here $\kappa_\sigma (\sigma = 1, \ldots, s')$ and $\epsilon_h \pm i\omega_h (h = 1, \ldots, s'')$, where $s' + 2s'' = r$ are the roots of the characteristic equation

$$\Delta(p) = 0 \tag{1.21}$$

The multiplicities of the roots are denoted by q_σ and $q_{s'+h}$ respectively. $\Delta_\sigma(p)$ and $\Delta_{s'+h}(p)$ represent the polynomials

$$\Delta_\sigma(p) = \frac{\Delta(p)}{(p - \varkappa_\sigma)^{q_\sigma}}, \qquad \Delta_{s'+h}(p) = \frac{\Delta(p)}{(p - \varepsilon_h - i\omega_h)^{q_{s'+h}}} \tag{1.22}$$

The prime on the summation sign in (1.20) indicates that the term under the summation sign refers not to an isolated root, but to the whole group of coincident roots of the characteristic equation (1.21). Using the theorem of multiplication of representations, and in accordance with (1.18) and (1.14), we have

$$\frac{\Phi(p)\,\Xi(p)}{\Delta(p)} \doteq \int_0^t N(t - \tau) X(\tau)\, d\tau \tag{1.23}$$

Thus, in accordance with (1.17), (1.18) and (1.23) the solution of the matrix differential equation (1.12) is

$$z(t) = N(t) z(0) + \int_0^t N(t - \tau) X(\tau)\, d\tau \tag{1.24}$$

Since the functions $X_\mu(t)$ for which $\mu \neq \sigma_l$ $(l = 1, \ldots, n)$ vanish identically, it follows that the elements of the matrix z will be

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t) z_k(0) + \int_0^t \sum_{i=1}^{n} N_{j\sigma_i}(t - \tau) X_{\sigma_i}(\tau)\, d\tau \qquad (j = 1, \ldots, r) \tag{1.25}$$

or, in accordance with (1.7)

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t)\, z_k(0) + \int_0^t \sum_{l=1}^{n} \sum_{i=1}^{n} N_{j\sigma_i}(t-\tau)\,\frac{B_{li}}{\Delta^*}\, x_l(\tau)\, d\tau \qquad (j = 1, \ldots, r) \tag{1.26}$$

Defining the function $W_{jl}(t)$ by

$$W_{jl}(t) = \sum_{i=1}^{n} N_{j\sigma_i}(t)\,\frac{B_{li}}{\Delta^*} \qquad \begin{pmatrix} j = 1, \ldots, r \\ l = 1, \ldots, n \end{pmatrix} \tag{1.27}$$

the general solution of (1.10) can be written down in the form

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t)\, z_k(0) + \sum_{l=1}^{n} \int_0^t W_{jl}(t-\tau)\, x_l(\tau)\, d\tau \qquad (j = 1, \ldots, r) \tag{1.28}$$

Let us now consider the determination of the position of the system in phase space in the case where the law obeyed by the external forces $x_l(t)$ ($l = 1, \ldots, n$) is known but the initial values of the phase coordinates $z_k(0)$ ($k = 1, \ldots, r$) are unknown.

We shall assume that only one of the phase coordinates z_s can be measured and the origin for this coordinate is unknown.

Let us choose a new arbitrary origin and measure the deviations $S(t_1)$, $S(t_2)$, ..., $S(t_{r+1})$ of the phase coordinate z_s from the new origin at times $t_1, \ldots, t_{r+1}$. Since

$$S(t_i) = S^* + z_s(t_i) \qquad (i = 1, \ldots, r+1) \tag{1.29}$$

where S^* is the deviation of the new origin from the original origin, and using the notation

$$S(t_{\mu+1}) - S(t_\mu) = L_\mu \qquad (\mu = 1, \ldots, r) \tag{1.30}$$

we are led to the following relation between the changes in the phase coordinate z_s and the results of measurement L_μ:

$$z_s(t_{\mu+1}) - z_s(t_\mu) = L_\mu \qquad (\mu = 1, \ldots, r) \tag{1.31}$$

This expression does not contain S^*.

Substituting into (1.31) the values of $z_s(t_{\mu+1})$ and $z_s(t_\mu)$ as given by (1.28), we obtain the following system of linear algebraic equations in terms of the initial values $z_k(0)$ of the phase coordinates

$$\sum_{k=1}^{r} [N_{sk}(t_{\mu+1}) - N_{sk}(t_\mu)]\, z_k(0) = \qquad (\mu = 1, \ldots, r) \tag{1.32}$$

$$= L_\mu - \sum_{l=1}^{n} \int_0^{t_{\mu+1}} W_{sl}(t_{\mu+1} - \tau)\, x_l(\tau)\, d\tau + \sum_{l=1}^{n} \int_0^{t_\mu} W_{sl}(t_\mu - \tau)\, x_l(\tau)\, d\tau$$

Having determined the initial values $z_k(0)$ from (1.32), we can use (1.28) to find the values of $z_j(t)$ for any time t.

2. Non-steady-state systems. The equations of motion of a non-steady-state controlled system

$$\sum_{k=1}^{n} f_{jk}(D)\, y_k = x_j(t) \qquad (j = 1, \ldots, n) \tag{2.1}$$

differ from (1.1) only in the fact that the coefficients of the polynomials $f_{jk}(D)$ will no longer be constants but, instead, certain given functions of time. The phase coordinates z_j defined by (1.5) will now satisfy the following system of differential equations with variable coefficients

$$\dot{z}_j + \sum_{k=1}^{r} a_{jk}(t)\, z_k = X_j(t) \qquad (j = 1, \ldots, r) \tag{2.2}$$

which can be derived similarly to (1.10).

The solution of (2.2) is of the form [4]

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t, 0)\, z_k(0) + \sum_{l=1}^{n} \int_0^t W_{jl}(t, \tau)\, x_l(\tau)\, d\tau \qquad (j = 1, \ldots, r) \tag{2.3}$$

Here

$$W_{jl}(t, \tau) = \sum_{i=1}^{n} N_{j\sigma_i}(t, \tau) \frac{B_{li}(\tau)}{\Delta^*(\tau)} \qquad \begin{pmatrix} j = 1, \ldots, r \\ l = 1, \ldots, n \end{pmatrix} \tag{2.4}$$

and $N_{jk}(t, \tau)$ are the elements of the matrix $N(t, \tau) = \theta(t)\theta^{-1}(\tau)$, where $\theta(t)$ is the fundamental matrix of the homogeneous matrix equation which can be obtained from (2.2) with $X_j(t) \equiv 0$ $(j = 1, \ldots, r)$.

Similarly to (1.32), the initial values $z_k(0)$ of the phase coordinates can be calculated from the results of measurements described above with the aid of the following set of linear algebraic equations:

$$\sum_{k=1}^{r} [N_{sk}(t_{\mu+1}, 0) - N_{sk}(t_\mu, 0)]\, z_k(0) = L_\mu - \sum_{l=1}^{n} \int_0^{t_{\mu+1}} W_{sl}(t_{\mu+1}, \tau)\, x_l(\tau)\, d\tau +$$
$$+ \sum_{l=1}^{n} \int_0^{t_\mu} W_{sl}(t_\mu, \tau)\, x_l(\tau)\, d\tau \qquad (\mu = 1, \ldots, r) \tag{2.5}$$

In order to determine the functions $W_{sl}(t_\zeta, \tau)$ where t_ζ is a fixed quantity, it is necessary to have a knowledge of $N_{s\xi}(t_\zeta, \tau)$ which represent the elements of the matrix weight function $N(t, \tau)$ for $t = t_\zeta$.

These elements are given by

$$N_{s\xi}(t_\zeta, \tau) = Z_\xi(\tau) \qquad (2.6)$$

where $Z_\xi(\tau)$ are the integrals of

$$\frac{dZ_\xi}{d\tau} - \sum_{k=1}^{r} a_{k\xi}(\tau) Z_k = 0 \qquad (\xi = 1, \ldots, r) \qquad (2.7)$$

which assume the following values at $t = t_\zeta$:

$$Z_s(t_\zeta) = 1, \quad Z_k(t_\zeta) = 0 \qquad (k = 1, \ldots, s-1, s+1, \ldots, r) \qquad (2.8)$$

Thus, in order to determine $N_{sk}(t_{\mu+1}, 0)$, $N_{sk}(t_\mu, 0)$ and the functions $W_{sl}(t_{\mu+1}, \tau)$, $W_{sl}(t_\mu, \tau)$ $(\mu = 1, \ldots, r)$, it is necessary to integrate (2.7) $(r + 1)$ times, with t_ζ in (2.8) given by $t_\zeta = t_1, \ldots, t_{r+1}$.

Having found the initial values $z_k(0)$ of the phase coordinates, one can, with the aid of (2.3), determine the position of the system in phase space for any given instant of time t^*. This involves a preliminary calculation of the weight functions $N_{jk}(t^*, \tau)$ for $j = 1, \ldots, r$, which in turn involves the integration of (2.7) r times with t_ζ and s in (2.8) given by $t_\zeta = t^*$, and $s = 1, \ldots, r$, respectively. The method for the solution of the above problem with the aid of electronic computers also follows from the above analysis.

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